Systems & Control Letters 7 (1986) 25–34
North-Holland

February 1986

A note on the structure of two subsets of the parameter space in adaptive control problems

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Received 3 August 1985
Revised 3 October 1985

Abstract: We study the geometric structure of two subsets of the parameter space that are of interest in the context of adaptive LQ-control. The first set can be considered as the set of possible limit points of an adaptive control algorithm, whereas the second can be seen as the set of desirable limit points. Our main result is that these sets are C^{ω}-manifolds.

AMS Subject Classification: 93C40.

Keywords: Adaptive LQ control, C^{ω}-manifold.

1. Introduction

In adaptive control one is faced with the problem of regulating a plant of which not all the characteristics are known.

One way of attacking this problem is the following: Assume that the plant is described by a member of a set of models, for each element of which one knows exactly how to control the corresponding system in order to achieve a certain desired behaviour. Then based on the observed data (coming from the plant) one tries to choose an element from the model set which explains the observed data 'best'. Then one acts as if this element represents the plant.

This procedure is meant to be done 'on line', and could be considered as a continuing alternation of estimation and control.

Now the question arises whether this procedure does what it is supposed to do, namely forcing the plant to behave as desired. What one means by desired depends upon the particular situation, but one can think of parameter identification, optimal closed-loop behaviour, identification of some control law, behaviour according to some reference model, etc. All this one may wish to achieve in finite time or asymptotically, with probability one, in expectation or in whatever sense one might think of.

In all these situations two subsets of the model set show up in a natural way. The first set, we call it H, is the set of all models that are equivalent with the real system (which is supposed to be a member of the model set) in the sense that they all lead to the same (controlled) behaviour as the desired behaviour of the real system.

The second set, G, consists of those models that on the basis of the behaviour of the plant cannot be distinguished from each other, due to the experimental circumstances (for instance identification in closed loop).

Now it could happen that an adaptive control scheme as described above converges to an element in G which does not correspond to desired behaviour (i.e. does not belong to H). This problem has been observed for instance in a paper by Borkar and Varaiya [2].

It may also happen that G is contained in H in which case this problem cannot occur; an example of this situation is studied in a paper by Becker, Kumar and Wei [1].

In a paper by Lin, Kumar and Seidman a study is made of the situation where the state space is one dimensional, the cost criterion is quadratic and a specific identification algorithm is used [7]. The possible limit points of this algorithm are investigated using the 'ordinary differential equation method' as developed by L. Ljung.

In this note we will study the two sets in the following situation. The model class will be a generic subset of the set of all linear time-invariant n-dimensional systems with m inputs and p outputs. The integer m, n and p, as well as the output matrix are supposed to be known. The desired behaviour will be dictated by a quadratic cost criterion. We emphasize that the definition and the relevance of these two sets are independent of how at each time instant a particular model is selected from the model set, and therefore we will not refer to any estimation algorithm.

In [8] the intersection of the two sets is studied; here we restrict our attention to the geometric, algebraic and topological structure of G and H separately. We will treat both the continuous and the discrete time case. The results for these two cases are identical geometrically, but algebraically and topologically, they are different.

The organization of the paper is as follows. First we recall some preliminaries which we will need, secondly we will give the problem statement. We will then state and prove a geometric result. Next we will give some algebraic and topological properties of the set H in the continuous-time case and finally we will give some simple but illustrative examples.

2. Preliminaries

We recall one of the equivalent definitions that could be given of an m-dimensional C^k-manifold in $\mathbb{R}^n$. See [9].

Definition 2.1. Let $X \subseteq \mathbb{R}^n$. X is an embedded m-dimensional C^k-manifold, if $\forall \tilde{x} \in X$, $\exists U \subseteq \mathbb{R}^n$, open, with $\tilde{x} \in U$ and a C^k-function $L: U \to \mathbb{R}^{n-m}$ such that:

(i) $L(\tilde{x}) = 0$.

(ii) $L^{-1}(\{0\}) = X \cap U$.

(iii) The derivative of L with respect to x, evaluated in $\tilde{x}$, has full rank.

Lemma 2.1. *Let M, $N \in \mathbb{R}^{p\times q}$, define $[M,N] := \mathrm{Tr}(MN^{\mathrm{T}})$. This defines an inner product.* (Tr *denotes the trace of a matrix.*)

Lemma 2.2. *Let $(X,[\cdot,\cdot]_X)$ and $(Y,[\cdot,\cdot]_Y)$ be finite-dimensional inner product spaces, and $F: X \to Y$ a linear map.*

(i) *There exists one and only one linear map $F^*: Y \to X$ such that for all $x \in X$ and for all $y \in Y$, $[Fx, y]_Y = [x, F^*y]_X$. F^* is called the adjoint operator of F with respect to $[\cdot,\cdot]_X$ and $[\cdot,\cdot]_Y$.*

(*ii*) *F is surjective iff F^* is injective.*

Proof. See [4].

Lemma 2.3. *Let M, $N \in \mathbb{R}^{p\times p}$, let Λ_1, $\Lambda_2: \mathbb{R}^{p\times p} \to \mathbb{R}^{p\times p}$ be defined by $\Lambda_1(X) = X - M^{\mathrm{T}}XN$ and $\Lambda_2(X) = M^{\mathrm{T}}X + XN$. Then ($\sigma$ denotes spectrum):*

(i) $\sigma(\Lambda_1) = 1 - \sigma(M)\times\sigma(N) = \{1 - \lambda\mu \mid \lambda \in \sigma(M),\ \mu \in \sigma(N)\}$.

(ii) $\sigma(\Lambda_2) = \sigma(M) + \sigma(N) = \{\lambda + \mu \mid \lambda \in \sigma(M),\ \mu \in \sigma(N)\}$.

Proof. See [6].

3. A geometric result

Let $n,m,p \in \mathbb{N}$, $n \geqslant m$, $C \in \mathbb{R}^{p\times n}$, fixed. Define $E \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ by

$$E := \{(A, B) \mid A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, (A, B, C) \text{ minimal, } A \text{ invertible, } B \text{ of full rank}\}$$

and

$$P := \{ K \in \mathbb{R}^{n \times n} \mid K = K^{\mathrm{T}} > 0 \}.$$

P is obviously a $\frac{1}{2}n(n+1)$-dimensional C^{ω}-manifold in $\mathbb{R}^{n \times n}$.

Remark. The invertibility of the A-matrix is used in the discrete-time case only.

Consider the linear systems

$$\dot{x} = Ax_t + Bu_t, \quad x_0 \in \mathbb{R}^n \quad \text{(continuous-time case)},$$

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 \in \mathbb{R}^n \quad \text{(discrete-time case)},$$

where $(A, B) \in E$, and suppose we want to choose u_t and u_k such that the following expressions are minimized:

$$J_{\mathrm{c}} = \int_0^{\infty} \left(x_t^{\mathrm{T}} Q x_t + u_t^{\mathrm{T}} R u_t \right) \mathrm{d}t, \qquad J_{\mathrm{d}} = \sum_{k=0}^{\infty} \left(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right)$$

where $Q = C^{\mathrm{T}}C$ and $R = R^{\mathrm{T}} > 0$ (subscripts c and d refer to continuous and discrete time respectively).

The solutions of these problems are well known (see [5]), and are given by

$$u_t = F_{\mathrm{c}}(A, B)x_t, \qquad u_k = F_{\mathrm{d}}(A, B)x_k,$$

where

$$F_{\mathrm{c}}(A, B) = -R^{-1}B^{\mathrm{T}}K_{\mathrm{c}}, \qquad F_{\mathrm{d}}(A, B) = -\left(B^{\mathrm{T}}K_{\mathrm{d}}B + R\right)^{-1}B^{\mathrm{T}}K_{\mathrm{d}}A,$$

and K_{c} and K_{d} are the unique solutions within P of

$$A^{\mathrm{T}}K + KA - KBR^{-1}B^{\mathrm{T}}K + Q = 0, \qquad \text{(CARE)}$$

$$K - A^{\mathrm{T}}KA + A^{\mathrm{T}}KB(B^{\mathrm{T}}KB + R)^{-1}B^{\mathrm{T}}KA - Q = 0, \qquad \text{(DARE)}$$

respectively.

Suppose the plant is represented by a fixed pair $(A_0, B_0) \in E$. Define the following subsets of E:

$$G_{\mathrm{c}} := \{ (A, B) \in E \mid A + BF_{\mathrm{c}}(A, B) = A_0 + B_0 F_{\mathrm{c}}(A, B) \},$$

$$H_{\mathrm{c}} := \{ (A, B) \in E \mid F_{\mathrm{c}}(A, B) = F_{\mathrm{c}}(A_0, B_0) \},$$

and G_{d} and H_{d} similarly.

The interpretation of these sets is the following: The set G can be considered as the invariant region of the parameter space under the use of an adaptive control scheme in the following sense: If we take a pair $(A, B) \in E$, and for some reason we think that (A, B) represents the real system, then we will apply the feedback $F(A, B)$. The resulting closed-loop system will then be $A_0 + B_0F(A, B)$ (because the real system is (A_0, B_0)). We expect however the closed-loop system to be $A + BF(A, B)$, so if $(A, B) \in G$ then the state sequence and the input sequence of $A_0 + B_0F(A, B)$ and $A + BF(A, B)$ will be exactly the same! So we will never change our minds about the system parameters.

G can also be seen as the set of possible limit points of an adaptive control scheme.

H is the set of all pairs (A, B) for which the optimal control law is exactly the control law one is looking for (i.e. the optimal control law belonging to the plant). If the goal is to identify only this control law rather than (A_0, B_0), H could be seen as the set of desirable limit points of the adaptive control scheme.

The main result we have is:

Theorem 3.1. (i) *H_c and H_d are $(n \times n)$-dimensional C^ω-manifolds.*
(ii) *G_c and G_d are $(m \times n)$-dimensional C^ω-manifolds.*

In order to prove this theorem we will first derive a result that is interesting in its own right. Rather than giving detailed proofs for both the continuous and the discrete time case, we restrict our attention to one of these two cases as far as full proofs are concerned, and we will only point out the major steps in the other case. The discrete-time calculations are somewhat more complicated because of the denominator $(B^TK_dB + R)$ in the equations for F_d and K_d.

Lemma 3.2. *There are C^ω-functions K_c and $K_d: E \to P$ such that $K_c(A, B)$ and $K_d(A, B)$ satisfy* CARE *and* DARE *respectively, for all $(A, B) \in E$.*

Proof (discrete time). A proof for the continuous-time case can be found in [3], we give the proof for the discrete-time case for the sake of completeness.

The implicit function theorem will be used to get the result. Define $L_d: E \times P \to \mathbb{R}^{n(n+1)/2}$ by

$$L_d(A, B, K) := K - A^TKA + A^TKB(B^TKB + R)^{-1}B^TKA - Q.$$

Since $R > 0$, L_d is C^ω. Note that $\forall(\tilde{A}, \tilde{B}) \in E$, $L_d(\tilde{A}, \tilde{B}, \tilde{K}) = 0$ where $\tilde{K}$ is the solution within P of DARE. We will now calculate the derivitative of L_d with respect to K, evaluated in such a triple $(\tilde{A}, \tilde{B}, \tilde{K})$. This will be a linear map $\Lambda_d: \mathbb{R}^{n(n+1)/2} \to \mathbb{R}^{n(n+1)/2}$ of which the action on $\Delta K \in \mathbb{R}^{n(n+1)}$ can be found by the following calculation (we will use the private notation $\overset{L}{=}$ to denote equality as far as linear terms in the 'Δ-variable(s)' are concerned):

$$\begin{aligned}
\Lambda_d(\Delta K) &\overset{L}{=} L_d(\tilde{A}, \tilde{B}, \tilde{K} + \Delta K) \\
&\overset{L}{=} \tilde{K} + \Delta K - \tilde{A}^T(\tilde{K} + \Delta K)\tilde{A} + \tilde{A}^T(\tilde{K} + \Delta K)\tilde{B}(\tilde{B}^T(\tilde{K} + \Delta K)\tilde{B} + R)^{-1}\tilde{B}^T(\tilde{K} + \Delta K)\tilde{A} - Q \\
&\overset{L}{=} \Delta K - \tilde{A}^T\Delta K\tilde{A} \\
&\quad + \tilde{A}^T(\tilde{K} + \Delta K)\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\left(\sum_{j=0}^{\infty}(-1)^j\left[\tilde{B}^T\Delta K\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\right]^j\right)\tilde{B}^T(\tilde{K} + \Delta K)\tilde{A} \\
&\overset{L}{=} \Delta K - \tilde{A}^T\Delta K\tilde{A} + \tilde{A}^T\Delta K\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\tilde{B}^T\tilde{K}\tilde{A} + \tilde{A}^T\tilde{K}\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\tilde{B}^T\Delta K\tilde{A} \\
&\quad - \tilde{A}^T\tilde{K}\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\tilde{B}^T\Delta K\tilde{B}(\tilde{B}^T\tilde{K}\tilde{B} + R)^{-1}\tilde{B}^T\tilde{K}\tilde{A} \\
&\overset{L}{=} \Delta K - \left\{(\tilde{A} + \tilde{B}F_d(\tilde{A}, \tilde{B}))^T\Delta K(\tilde{A} + \tilde{B}F_d(\tilde{A}, \tilde{B}))\right\}.
\end{aligned}$$

Since $\tilde{A} + \tilde{B}F_d(\tilde{A}, \tilde{B})$ is strictly stable (see [5]), it follows by Lemma 2.3(i) that $0 \notin \sigma(\Lambda_d)$, hence Λ_d is non-singular.

Now the implicit function theorem yields the existence of the function K_d in a neighbourhood of $(\tilde{A}, \tilde{B})$. Since $(\tilde{A}, \tilde{B})$ was arbitrary and the solution of DARE is unique (within P), K_d is well defined on E.

Corollary 3.3. *F_c and F_d are C^ω-functions on E.*

Proof. This is immediate from the facts that F_c and F_d are C^ω-functions of (A, B, K) and the previous lemma.

For the proof of 3.1(i) (discrete-time case) we will need the following lemma:

Lemma 3.4. *For all* $(A, B) \in E$ *we have that* $(A + BF(A, B))$ *is non-singular.*

Proof. Suppose $x_0 \in \operatorname{Ker}(A + BF_d(A, B))$; then $x_k = 0$ and $u_k = 0$, for all $k \geqslant 1$. Hence

$$\begin{aligned} x_0^T K_d x_0 &= x_0^T Q x_0 + u_0^T R u_0 \quad \text{(since the optimal costs are given by } x_0^T K x_0\text{)} \\ &= x_0^T\big(Q + F_d(A, B)^T R F_d(A, B)\big)x_0 \\ &= x_0^T\big(K - A^T K(A + BF_d(A, B)) + F_d(A, B)^T R F_d(A, B)\big)x_0 \quad \text{(by DARE)}. \end{aligned}$$

This implies that $x_0^T F_d(A, B)^T R F_d(A, B) x_0 = 0$ and thus that $F_d(A, B)x_0 = 0$. Together with $(A + BF_d(A, B))x_0 = 0$ this gives $Ax_0 = 0$. Since by assumption A is non-singular, x_0 must be zero and hence $A + BF(A, B)$ is non-singular.

Proof of Theorem 3.1 (i) (discrete-time case). Define $\tilde{H}_d \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n(n+1)/2}$ by

$$\tilde{H}_d := \{(A, B, K) \mid (A, B) \in H_d,\ K = K_d(A, B) \in P\}.$$

Define $L: E \times P \to \mathbb{R}^{m\times n} \times \mathbb{R}^{n(n+1)/2}$ by

$$L(A, B, K) = (L_1(A, B, K), L_2(A, B, K)),$$

where

$$\begin{aligned} L_1(A, B, K) &= (B^T K B + R)^{-1} B^T K A + F_d(A_0, B_0), \\ L_2(A, B, K) &= K - A^T K A + K B(B^T K B + R)^{-1} B^T K A - Q. \end{aligned}$$

Note that $(A, B, K) \in \tilde{H}$ if and only if $L(A, B, K) = (0, 0)$, and that L is C^ω.

Fix a triple $(\tilde{A}, \tilde{B}, \tilde{K}) \in \tilde{H}_d$. We will calculate the derivative of L with respect to (A, B, K), evaluated in $(\tilde{A}, \tilde{B}, \tilde{K})$:

$$\begin{aligned} \Lambda_1(\Delta A, \Delta B, \Delta K) :&\overset{L}{=} L_1(\tilde{A} + \Delta A, \tilde{B} + \Delta B, \tilde{K} + \Delta K) \\ &\overset{L}{=} (\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1} B^T \tilde{K} \Delta A \\ &\quad + \big((\tilde{B} + \Delta B)^T (\tilde{K} + \Delta K)(\tilde{B} + \Delta B) + R\big)^{-1} (\tilde{B} + \Delta B)^T (\tilde{K} + \Delta K)\tilde{A} \\ &\overset{L}{=} (\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1} \Big\{ \tilde{B}^T \tilde{K} \Delta A + \sum_{j=0}^{\infty} (-1)^j \big[(\Delta B^T \tilde{K} \tilde{B} + \tilde{B}^T \Delta K \tilde{B} + \tilde{B}^T \tilde{K} \Delta B) \\ &\qquad \cdot (\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1}\big]^j (\tilde{B} + \Delta B)^T (\tilde{K} + \Delta K)\tilde{A} \Big\} \\ &\overset{L}{=} (\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1} \big\{ \tilde{B}^T \tilde{K} \Delta A + \Delta B^T \tilde{K} \tilde{A} + \tilde{B}^T \Delta K \tilde{A} \\ &\qquad - (\Delta B^T \tilde{K} \tilde{B} + \tilde{B}^T \Delta K \tilde{B} + \tilde{B}^T \tilde{K} \Delta B)(\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1} \tilde{B}^T \tilde{K} \tilde{A}) \big\} \\ &\overset{L}{=} (\tilde{B}^T \tilde{K} \tilde{B} + R)^{-1} \big\{ \tilde{B}^T \tilde{K} \Delta A + \Delta B^T \tilde{K} \tilde{A} + \tilde{B}^T \Delta K \tilde{A} + (\Delta B^T \tilde{K} \tilde{B} + \tilde{B}^T \Delta K \tilde{B} \\ &\qquad + \tilde{B}^T \tilde{K} \Delta B) F_d(\tilde{A}, \tilde{B}) \big\}. \end{aligned}$$

Similar calculations yield

$$\Lambda_2(\Delta A,\ \Delta B,\ \Delta K) :\overset{\mathrm{L}}{=} L_2(\tilde{A}+\Delta A,\ \tilde{B}+\Delta B,\ \tilde{K}+\Delta K)$$

$$\overset{\mathrm{L}}{=} -\left[\Delta A^{\mathrm{T}}\tilde{K}\tilde{A}+\tilde{A}^{\mathrm{T}}\tilde{K}\Delta A+\Delta A^{\mathrm{T}}\tilde{K}\tilde{B}F_{\mathrm{d}}(\tilde{A},\ \tilde{B})+F_{\mathrm{d}}(\tilde{A},\ \tilde{B})^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{K}\Delta A\right]$$

$$-\left[F_{\mathrm{d}}(\tilde{A},\ \tilde{B})^{\mathrm{T}}\Delta B^{\mathrm{T}}\tilde{K}\tilde{A}+\tilde{A}^{\mathrm{T}}\tilde{K}\Delta BF_{\mathrm{d}}(\tilde{A},\ \tilde{B})\right.$$

$$\left.+F_{\mathrm{d}}(\tilde{A},\ \tilde{B})^{\mathrm{T}}(\tilde{B}^{\mathrm{T}}\tilde{K}\Delta B+\Delta B^{\mathrm{T}}\tilde{K}\tilde{B})F_{\mathrm{d}}(\tilde{A},\ \tilde{B})\right]$$

$$+\Delta K-(\tilde{A}+\tilde{B}F_{\mathrm{d}}(\tilde{A},\ \tilde{B}))^{\mathrm{T}}\Delta K(\tilde{A}+\tilde{B}F_{\mathrm{d}}(\tilde{A},\ \tilde{B})).$$

Introduce $\tilde{C} := \tilde{A}+\tilde{B}F_{\mathrm{d}}(\tilde{A},\ \tilde{B})$, the optimal closed-loop matrix, and $\tilde{F}_{\mathrm{d}} := F_{\mathrm{d}}(\tilde{A},\ \tilde{B})$ to get more compact notation:

$$\Lambda_1(\Delta A,\ \Delta B,\ \Delta K)=(\tilde{B}^{\mathrm{T}}\tilde{K}\tilde{B}+R)^{-1}\{\tilde{B}^{\mathrm{T}}\tilde{K}\Delta A+\Delta B^{\mathrm{T}}\tilde{K}\tilde{C}+\tilde{B}^{\mathrm{T}}\tilde{K}\Delta BF_{\mathrm{d}}+\tilde{B}^{\mathrm{T}}\Delta K\tilde{C}\},$$

$$\Lambda_2(\Delta A,\ \Delta B,\ \Delta K)=-\left[\Delta A^{\mathrm{T}}\tilde{K}\tilde{C}+\tilde{C}^{\mathrm{T}}\tilde{K}\Delta A+\tilde{F}_{\mathrm{d}}^{\mathrm{T}}\Delta B^{\mathrm{T}}\tilde{K}\tilde{C}+\tilde{C}^{\mathrm{T}}\tilde{K}\Delta B\tilde{F}_{\mathrm{d}}\right]+\Delta K-\tilde{C}^{\mathrm{T}}\Delta K\tilde{C}.$$

We will now show that Λ is surjective. Define

$$\overline{\Lambda}=((\tilde{B}^{\mathrm{T}}\tilde{K}\tilde{B}+R)\Lambda_1,\ \Lambda_2)=(\overline{\Lambda}_1,\ \overline{\Lambda}_2);$$

the surjectiveness of $\overline{\Lambda}$ is equivalent with the surjectiveness of Λ, so we proceed with $\overline{\Lambda}$.

To show that $\overline{\Lambda}$ is surjective (or equivalently its matrix has full rank), it is by Lemma 2.2(ii) enough to show that its adjoint map (with respect to some inner product) is injective. The inner product given by 2.1 happens to be convenient for this purpose. In the computation of the adjoint map $\overline{\Lambda}$, we will gratefully use the basic facts that for any two matrices M, N of the same dimensions $\mathrm{Tr}(MN^{\mathrm{T}})=\mathrm{Tr}(N^{\mathrm{T}}M)$, and for any square matrix O: $\mathrm{Tr}(O)=\mathrm{Tr}(O^{\mathrm{T}})$.

Let $(U,\ V)\in\mathbb{R}^{n\times m}\times\mathbb{R}^{n(n+1)/2}$. Then

$$[\overline{\Lambda}(\Delta A,\ \Delta B,\ \Delta K),(U,\ V)]=\mathrm{Tr}(\overline{\Lambda}_1(\Delta A,\ \Delta B,\ \Delta K)U^{\mathrm{T}})+\mathrm{Tr}(\overline{\Lambda}_2(\Delta A,\ \Delta B,\ \Delta K)V^{\mathrm{T}})$$

$$=\mathrm{Tr}(\tilde{B}^{\mathrm{T}}\tilde{K}\Delta AU^{\mathrm{T}})+\mathrm{Tr}(\Delta B^{\mathrm{T}}\tilde{K}\tilde{C}U^{\mathrm{T}})$$

$$+\mathrm{Tr}(\tilde{B}^{\mathrm{T}}\tilde{K}\Delta B\tilde{F}_{\mathrm{d}}U^{\mathrm{T}})+\mathrm{Tr}(\tilde{B}^{\mathrm{T}}\Delta K\tilde{C}U^{\mathrm{T}})-\mathrm{Tr}(\Delta A^{\mathrm{T}}\tilde{K}\tilde{C}V^{\mathrm{T}})$$

$$-\mathrm{Tr}(\tilde{C}^{\mathrm{T}}\tilde{K}\Delta AV^{\mathrm{T}})-\mathrm{Tr}(\tilde{F}_{\mathrm{d}}^{\mathrm{T}}\Delta B^{\mathrm{T}}\tilde{K}\tilde{C}V^{\mathrm{T}})-\mathrm{Tr}(\tilde{C}^{\mathrm{T}}\tilde{K}\Delta B\tilde{F}_{\mathrm{d}}V^{\mathrm{T}})$$

$$+\mathrm{Tr}(\Delta KV^{\mathrm{T}})-\mathrm{Tr}(\tilde{C}^{\mathrm{T}}\Delta K\tilde{C}V^{\mathrm{T}})$$

$$=\mathrm{Tr}(\Delta AU^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{K})+\mathrm{Tr}(\Delta BU\tilde{C}^{\mathrm{T}}\tilde{K})$$

$$+\mathrm{Tr}(\Delta B\tilde{F}_{\mathrm{d}}U^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{K})+\mathrm{Tr}(\Delta K\tilde{C}U^{\mathrm{T}}\tilde{B}^{\mathrm{T}})-\mathrm{Tr}(\Delta AV\tilde{C}^{\mathrm{T}}\tilde{K})$$

$$-\mathrm{Tr}(\Delta AV^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{K})-\mathrm{Tr}(\Delta B\tilde{F}_{\mathrm{d}}V\tilde{C}^{\mathrm{T}}\tilde{K})-\mathrm{Tr}(\Delta B\tilde{F}_{\mathrm{d}}V^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{K})$$

$$+\mathrm{Tr}(\Delta KV^{\mathrm{T}})-\mathrm{Tr}(\Delta K\tilde{C}V^{\mathrm{T}}\tilde{C}^{\mathrm{T}})$$

$$=\mathrm{Tr}(\Delta A(U^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{K}-V\tilde{C}^{\mathrm{T}}\tilde{K}-V^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{K}))$$

$$+\mathrm{Tr}(\Delta B(U\tilde{C}^{\mathrm{T}}\tilde{K}+\tilde{F}_{\mathrm{d}}U^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{K}-\tilde{F}_{\mathrm{d}}V\tilde{C}^{\mathrm{T}}\tilde{K}-\tilde{F}_{\mathrm{d}}V^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{K}))$$

$$+\mathrm{Tr}(\Delta K(\tilde{C}U^{\mathrm{T}}\tilde{B}+V^{\mathrm{T}}-\tilde{C}V^{\mathrm{T}}\tilde{C}^{\mathrm{T}})).$$

Hence by Lemma 2.2(i),

$$\overline{\Lambda}^*(U,\ V)=(\tilde{K}\tilde{B}U-\tilde{K}\tilde{C}V^{\mathrm{T}}-\tilde{K}\tilde{C}V,\ \tilde{K}\tilde{C}U^{\mathrm{T}}+\tilde{K}\tilde{B}U\tilde{F}_{\mathrm{d}}^{\mathrm{T}}-\tilde{K}\tilde{C}V^{\mathrm{T}}\tilde{F}_{\mathrm{d}}^{\mathrm{T}}-\tilde{K}\tilde{C}V\tilde{F}_{\mathrm{d}}^{\mathrm{T}},\ \tilde{B}^{\mathrm{T}}U\tilde{C}^{\mathrm{T}}+V-\tilde{C}V\tilde{C}^{\mathrm{T}}).$$

To show that $\bar{\Lambda}^*$ is injective, we just put $\bar{\Lambda}^*(U, V) = (0, 0, 0)$, this gives the following equations:

$$E_1: \quad \tilde{K}\tilde{B}U - \tilde{K}\tilde{C}V^{\mathrm{T}} - \tilde{K}\tilde{C}V = 0,$$

$$E_2: \quad \tilde{K}\tilde{C}U^{\mathrm{T}} + \tilde{K}\tilde{B}U\tilde{F}_{\mathrm{d}}^{\mathrm{T}} - \tilde{K}\tilde{C}V^{\mathrm{T}}\tilde{F}_{\mathrm{d}}^{\mathrm{T}} - \tilde{K}\tilde{C}V\tilde{F}_{\mathrm{d}}^{\mathrm{T}} = 0,$$

$$E_3: \quad \tilde{B}^{\mathrm{T}}U\tilde{C}^{\mathrm{T}} + V - \tilde{C}V\tilde{C}^{\mathrm{T}} = 0.$$

$E_2 - E_1\tilde{F}_{\mathrm{d}}^{\mathrm{T}}$ gives $\tilde{K}\tilde{C}U^{\mathrm{T}} = 0$.

Since $\tilde{A}$ is invertible, it follows from Lemma 3.4 that $\tilde{C}$ is invertible, moreover $\tilde{K} > 0$ from which we can conclude that $U = 0$. Then E_3 becomes

$$V - \tilde{C}V\tilde{C}^{\mathrm{T}} = 0.$$

Since $\tilde{C}$ is strictly stable it follows from Lemma 2.3(i) that $V = 0$, showing that $\bar{\Lambda}^*$ is injective and thus that $\bar{\Lambda}$ is surjective and hence Λ is surjective.

Now all the conditions of Definition 2.1 are fulfilled, hence $\tilde{H}_{\mathrm{d}}$ is an $n \times n$-dimensional C^{ω}-manifold in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n(n+1)/2}$. Since $\tilde{K}$ depends C^{ω} on $(\tilde{A}, \tilde{B})$, it is easy to see that H_{d} is an $n \times n$-dimensional C^{ω}-manifold in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. This completes the proof of part (i).

The proof for the continuous-time case is completely analogous.

Proof of Theorem 3.1(ii) (continuous-time case). The proof goes along the same lines as the proof of part (i). We will give the proof for the continuous-time case only. The discrete-time case is completely analogous, but technically more involved.

Define $\tilde{G}_{\mathrm{c}} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n(n+1)/2}$ by

$$\tilde{G}_{\mathrm{c}} := \{(A, B, K) \mid (A, B) \in G_{\mathrm{c}}, K = K_{\mathrm{c}}(A, B)\}.$$

Define $L: E \times P \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n(n+1)/2}$ by

$$L(A, B, K) = (L_1(A, B, K), L_2(A, B, K)),$$

where

$$L_1(A, B, K) = A - BB^{\mathrm{T}}R^{-1}K - A_0 + B_0B^{\mathrm{T}}R^{-1}K, \qquad L_2(A, B, K) = A^{\mathrm{T}}K + KA - KBR^{-1}B^{\mathrm{T}}K + Q.$$

Note that $(A, B, K) \in \tilde{G}_{\mathrm{c}}$ if and only if $L(A, B, K) = (0, 0)$, and that L is C^{ω}.

Fix a triple $(\tilde{A}, \tilde{B}, \tilde{K}) \in G_{\mathrm{c}}$. The derivative of L with respect to (A, B, K), evaluated in $(\tilde{A}, \tilde{B}, \tilde{K})$, is

$$\begin{aligned}\Lambda(\Delta A, \Delta B, \Delta K) &= (\Lambda_1(\Delta A, \Delta B, \Delta K)), \Lambda_2(\Delta A, \Delta B, \Delta K)) \\ &= \big(\Delta A - \Delta B\tilde{B}^{\mathrm{T}}R^{-1}\tilde{K} - \tilde{B}\Delta B^{\mathrm{T}}R^{-1}\tilde{K} - \tilde{B}\tilde{B}^{\mathrm{T}}R^{-1}\Delta K + B_0\Delta B^{\mathrm{T}}R^{-1}\tilde{K} + B_0\tilde{B}^{\mathrm{T}}R^{-1}\Delta K, \\ &\quad \Delta A^{\mathrm{T}}\tilde{K} + \tilde{A}^{\mathrm{T}}\Delta K + \Delta K\tilde{A} + \tilde{K}\Delta A - \Delta K\tilde{B}R^{-1}\tilde{B}^{\mathrm{T}}\tilde{K} \\ &\quad - \tilde{K}\Delta BR^{-1}\tilde{B}^{\mathrm{T}}\tilde{K} - \tilde{K}\tilde{B}R^{-1}\Delta B^{\mathrm{T}}\tilde{K} - \tilde{K}\tilde{B}R^{-1}\tilde{B}^{\mathrm{T}}\Delta K\big).\end{aligned}$$

Now using the same method as in part (i), one can show that the adjoint of Λ is given by

$$\begin{aligned}\Lambda^*(U, V) = \big(&U + 2\tilde{K}V, - U\tilde{F}_{\mathrm{c}}^{\mathrm{T}} - \tilde{K}U^{\mathrm{T}}\tilde{B}R^{-1} + \tilde{K}U^{\mathrm{T}}B_0R^{-1} - 2\tilde{K}V\tilde{F}_{\mathrm{c}}^{\mathrm{T}}, \\ &- \tilde{B}R^{-1}\tilde{B}^{\mathrm{T}}U + \tilde{B}R^{-1}B_0^{\mathrm{T}}U + (\tilde{A} + \tilde{B}\tilde{F}_{\mathrm{c}})V + V(\tilde{A} + \tilde{B}\tilde{F}_{\mathrm{c}})^{\mathrm{T}}\big).\end{aligned}$$

It is easy to see that Λ^* is injective, hence Λ is surjective, showing that G_{c} is an $n \times m$-dimensional C^{ω}-manifold in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n(n+1)/2}$. As in part (i) it follows that G_{c} is an $n \times m$-dimensional manifold in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$.

4. Further characterization of H_c

Due to the absence of a denominator in CARE, we can give a complete parametrization of H_c.

Theorem 4.1.

$$H_c = \{(K^{-1}K_0A_0 + K^{-1}M,\ K^{-1}K_0B_0) \mid K \in P,\ M + M^T = 0\} \cap E.$$

Proof. It is a matter of verification that for every $K \in P$, and antisymmetric matrix M, the pair $(A, B) = (K^{-1}K_0A_0 + K^{-1}M,\ K^{-1}K_0B_0)$ satisfies

$$A^TK + KA - KBR^{-1}B^TK + Q = 0, \qquad -R^{-1}B^TK = -R^{-1}B_0^TK_0 (= F_0).$$

Suppose on the other hand that $(A, B) \in H_c$, and let $K \in P$ be the solution of CARE. Since

$$F(A, B) = -R^{-1}B^TK = F(A_0, B_0) = -R^{-1}B_0^TK_0,$$

it follows that

$$B = K^{-1}K_0B_0.$$

Now consider CARE, for both (A, B) and (A_0, B_0):

$$A^TK + KA - KBR^{-1}B^TK + Q = A^TK + KA - F_0^TRF_0 + Q = 0,$$

$$A_0^TK_0 + K_0A_0 - K_0B_0R^{-1}B_0^TK_0 + Q = A_0^TK_0 + K_0A_0 - F_0^TRF_0 + Q = 0.$$

Hence

$$A^TK + KA = A_0^TK_0 + K_0A_0. \qquad (*)$$

A particular solution of $(*)$ is given by

$$A = K^{-1}K_0A_0.$$

The solutions of the homogeneous equation, $X^TK + KX = 0$, are given by

$$X = K^{-1}M \quad \text{where } M + M^T = 0.$$

so $A = K^{-1}K_0A_0 + K^{-1}M$, for some antisymmetric matrix M. The proof is finished.

Remark. Theorem 3.1(i) (continuous time) follows also from Theorem 4.1. The proof of Theorem 4.1 is much simpler than that of Theorem 3.1, moreover it gives better insight into the structure of H_c.

One might conjecture that $(K^{-1}K_0A_0 + K^{-1}M,\ K^{-1}K_0B_0) \in E$ for all $K \in P$ and antisymmetric matrix M. This would then be a false conjecture, as will be shown in Example 3. However, we have the following:

Corollary 4.2. *The closure of H_c is connected.*

Proof. The set of (K, M) for which $(K^{-1}K_0A_0 + K^{-1}M,\ K^{-1}K_0B_0)$ is in E, is open and dense in the product space of P and the vector space of antisymmetric matrices because of the genericity of minimality. Hence $\overline{H}_c$ is the image of a connected set under a continuous map.

Remark. For the discrete-time case an analogous result has not been found, and in fact a counterexample to the connectedness of $\overline{H}_d$ will be given in the next section (Example 1).

5. Examples

Example 1. (Discrete time). Let $n = m = p = 1$, $r = 1$, $q = \frac{1}{2}$, $E = \{(a, b) \in \mathbb{R}^2 \mid a, b \neq 0\}$, $P = \{k \in \mathbb{R} \mid k > 0\}$, $a_0 = 1$, $b_0 = 1$. For this specific example $f_0 = f_d(a_0, b_0)$, appears to be $-\frac{1}{2}$. Hence

$$H_d = \{(a, b) \mid b \neq 0, f_d(a, b) = -\tfrac{1}{2}\}.$$

One may check that

$$H_d = \{(a, b) \mid a > 0,\ b^2 - ab - 2a^2 + 2 = 0\} \cup \{(a, b) \mid a < 0,\ b < 0,\ b^2 - ab - 2a^2 + 2 = 0\}.$$

For a picture see Figure 1.

Example 2. (Continuous time). In the one-dimensional state space case, H_c can easily be determined: From Section 4 it follows that for $(a_0, b_0) \in E$, H_c is given by

$$H_c = \{(k^{-1}k_0,\ k^{-1}k_0 b_0) \mid k > 0\}.$$

This implies that $b = ab_0/a_0$, where a ranges over all values that have the same sign as a_0.

Example 3. (Continuous time). There exist pairs (K, M) such that $(K^{-1}K_0A_0 + K^{-1}M,\ K^{-1}K_0B_0) \notin E$. Let $n = 2$, $m = p = r = 1$,

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} \frac{1}{3} & 1 \\ -3 & 0 \end{bmatrix}, \quad b_0 = \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}.$$

Calculations show that

$$K_0 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = [-2 \quad 0].$$

Take $K = K_0$, and

$$M = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}.$$

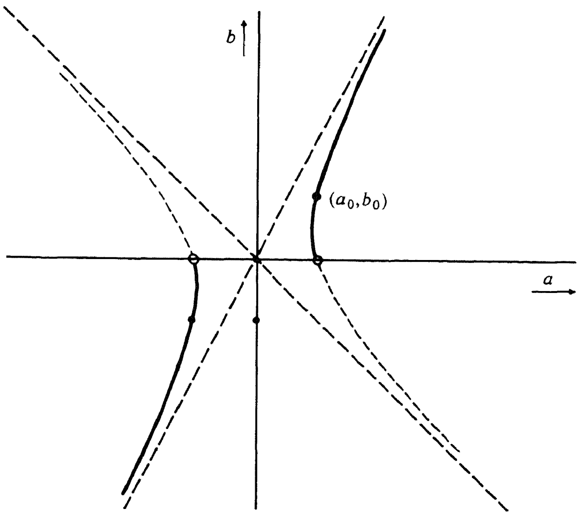

Fig. 1. H_d for $a_0 = b = 0 = 1$; $r = 1$, $q = \frac{1}{2}$ (H_d is the bold part of the picture).

Then

$$A=\begin{bmatrix}\frac{1}{3} & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix}\frac{2}{3}\\ 0\end{bmatrix}$$

which is obviously not in E.

Comment. The remarkable difference between discrete and continuous time is illustrated by Examples 1 and 2. H_d is part of a second-degree algebraic curve, where H_c is part of a linear curve. Furthermore Example 1 shows that H_d nor its closure is connected.

Example 3 shows that a parametrization of H_d in terms of *almost all* pairs (K, M) rather then *all* pairs, is the best we can achieve. However this does not imply that H_c is not connected.

6. Conclusions

In this paper we have investigated the structure of two subsets of a specific model class. For G and H_d we have derived a geometric result. For H_c we have also given a parametrization which gives more insight into the topological and algebraic structure.

Acknowledgement

I would like to thank P.R. Kumar for the stimulating discussions we had during his stay at the Centre for Mathematics and Computer Science.

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